HW8

(1) $\cos z = -\sin\left(z-\frac{\pi}{2}\right) = -\sum_{n=0}^{\infty} \frac{(-1)^n \left(z-\pi/2\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\left(z-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}$

Result follows by the unique of taylor series.

(2) $f = \sin(z^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (z^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+2}}{(2n+1)!}$

Therefore, $f^{(4n)}(0) = f^{(2n+1)}(0) = 0$.

(3) (a) $\sinh z = \frac{e^z - e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{1+2n}}{(1+2n)!}$

$\frac{\sinh z}{z^2} = \sum_{n=0}^{\infty} \frac{z^{-1+2n}}{(1+2n)!} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} + \frac{1}{z}$

(b) It follows from (a).

(4) For $0<|z|<4$, $\frac{1}{4z-z^2} = \frac{1}{4z}\left(\frac{1}{1-z/4}\right) = \frac{1}{4z}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n$

$\Rightarrow \frac{1}{4z-z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$.

(5) $f(z) = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{z^2}\right)^{2n+1} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{4n}}$

(6) $f(z) = \frac{1}{z} \cdot \frac{1}{1-\left(-\frac{1}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n}$

(7) In D_1, $f(z) = \frac{1}{z-1} - \frac{1}{z-2} = -\frac{1}{1-z} + \frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - 1\right) z^n$

In D_2, $f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} + \frac{1}{2}\frac{1}{1-\frac{z}{2}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=1}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$

D_3: $f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{z}\sum_{n=0}^{\infty}\frac{2^n}{z^n} = \sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n}$

(8) $\frac{z}{(z-1)(z-3)} = \frac{z-3+3}{(z-1)(z-3)} = \frac{3}{(z-1)(z-3)} + \frac{1}{z-1}$

$\frac{1}{z-3} = \frac{1}{z-1-2} = \frac{-1}{2}\frac{1}{1-\frac{z-1}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n}$

$\therefore \frac{z}{(z-1)(z-3)} = \frac{3}{z-1}\left(-\frac{1}{2}\right)\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n} + \frac{1}{z-1}$

$= -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{3}{2}\frac{1}{z-1} + \frac{1}{z-1} = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}$

9(a) $\frac{a}{z-a} = \frac{a}{z}\frac{1}{1-\frac{a}{z}} = \frac{a}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = \sum_{n=1}^{\infty}\frac{a^n}{z^n}$

(b) $\frac{a}{e^{i\theta}-a} = \frac{a}{\cos\theta - a + i\sin\theta} = \frac{a(\cos\theta - a - i\sin\theta)}{(\cos\theta-a)^2 + \sin^2\theta}$

$= \frac{a\cos\theta - a^2}{1-2a\cos\theta+a^2} - i\frac{a\sin\theta}{1-2a\cos\theta+a^2}$

$\sum_{n=1}^{\infty}\frac{a^n}{e^{i\theta n}} = \sum_{n=1}^{\infty} a^n(\cos n\theta - i\sin n\theta) = \sum_{n=1}^{\infty} a^n\cos n\theta - i\sum_{n=1}^{\infty} a^n \sin n\theta$

Compare the real parts and the imaginary parts, the result follows.

8) Refer to Ex. of tut 7.

10a)
$$J_n = \frac{1}{2\pi i}\int_C \frac{e^{\frac{z}{2}\left(w-\frac{1}{w}\right)}}{z^{n+1}}\, dz$$
$$= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{z/2\,(2i\sin\phi)}}{e^{(n+1)\phi i}}\, i e^{i\phi}\, d\phi$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\{-i\,(n\phi - z\sin\phi)\}}\, d\phi .$$

10b) It's due to the fact that $\cos\theta$ is even and sin is odd.

(11a) it comes from Laurent theorem directly.

~~(11b) $\cancel{f(\phi) = f(e^{i\phi})}$~~

(11b) $u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\,d\phi$

$$+\frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left\{2\cos(n(\theta-\phi))\right\}d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\,d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos\{n(\theta-\phi)\}\,d\phi$$